A Meditation on Patterns of Mathematical Notation

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Abstract

Mathematical ideas are usually expressed by written notations. Patterns in mathematical notation reflect patterns of ideas. The adjunction pattern is a sort of balancing act that is prominent in category theory. The recursion pattern is about repeated nesting, like Russian dolls. The mathematical theory of computation is also called recursion theory. This article is a series of thoughts, sometimes surprising examples, and theorems on these ubiquitous patterns.

Keywords: notation pattern, adjunction, recursion, education

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03D75 Abstract and axiomatic computability and recursion theory
03G30 Categorical logic

1 Introduction

There are patterns of algebraic notation repeated in diverse scientific and mathematical research disciplines. If geometry is, generally, about the invariants of symmetries in space, then symmetries of mathematical notations, written on flat sheets of paper, might lead to thinking that algebra is the geometry of notation. In any case, the mirror-like symmetry of the **adjunction pattern**

$$(F(A), B) = (A, G(B))$$

and the ouroborus symmetry of the **recursion pattern**

$$A = F(A)$$

recur frequently in mathematics and science. See Fig. (1).

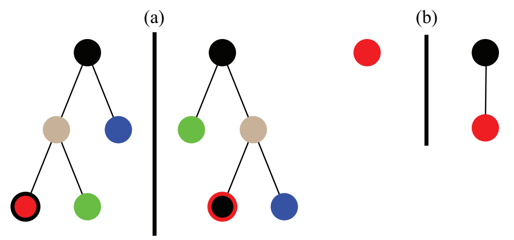

Figure 1: (a) The **adjunction pattern**. (b) The **recursion pattern**.

2 K-12 Patterns

Even in early mathematics education, K-12 students are routinely familiarized with use of numerical calculation rules, see Fig. (2). The **distribution pattern**

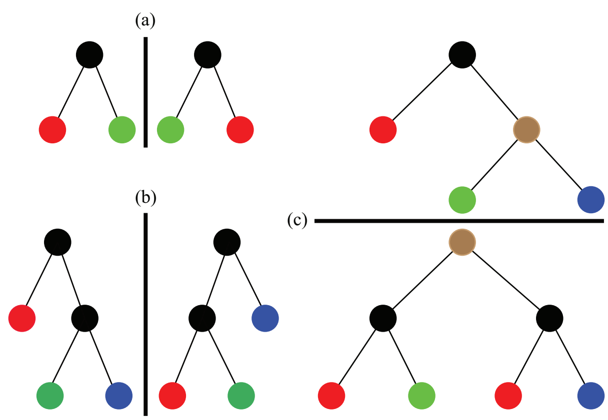

Figure 2: (a) **Commutation pattern** $AB = BA$. (b) **Association pattern** $A(BC) = (AB)C$. (c) **Distribution pattern** $A(B+C) = AB + AC$.

of multiplication with respect to addition is often explained with a geometrical diagram that amounts to saying "the whole is equal to the sum of its parts." The **distribution pattern** recurs in linear algebra: a function $U \xrightarrow{T} V$ of linear spaces over a field is a **linear transformation** if $T(x+y) = T(x) + T(y)$ (and secondarily $T(ax) = aT(x)$, which extends to all scalars a the formula $T(nx) = nT(x)$ for $n \in \mathbb{N}$, which follows by induction from the first formula). Also, the **distribution pattern** recurs centrally in category theory, since by definition a functor $C \xrightarrow{F} D$ of categories satisfies the **distribution pattern** $F(g \circ f) = F(g) \circ F(f)$ for composable maps f, g.

3 The Adjunction Pattern

An **adjunction pattern** is a sort of balancing act with multiple "moving parts."

Example 1. *(Linear Correlation) In undergraduate statistics courses, students learn an* ***adjunction pattern*** *without necessarily identifying it as such. Recall that hypothesis testing of linear correlation of two random variables may be performed with the "critical value method," which concludes by rejecting the null hypothesis if the absolute value of linear correlation r between the two random variables is greater than the critical value r_{crit}, which depends on the significance level and the sample size [49, pages 542–543][9, page 161]. Alternatively, the null hypothesis is rejected if the P-value is less than or equal to the significance level. In other words, there is a logical equivalence*

$$P - value(r, \delta) < \alpha \Leftrightarrow |r| \leq r_{crit}(\alpha, \delta)\,. \tag{1}$$

The moving parts are r, r_{crit}, α and δ. To emphasize the notational symmetry in this ***adjunction pattern****, it may be re-presented as*

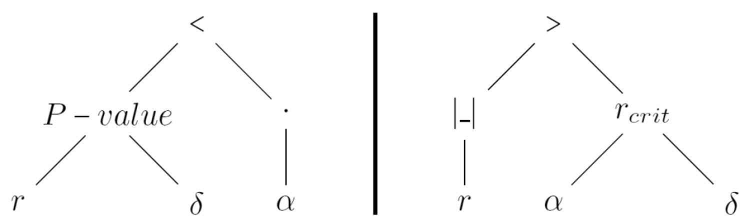

Figure 3: Hypothesis Test for linear correlation.

Example 2. *(Set Theory) For a set A, $(A \times \_)$ denotes the Cartesian product operator defined by $(A \times \_)(X) = A \times X = \{\,(a, x) \mid a \in A \wedge x \in X\,\}$. Also, the exponentiation operator $(A\hat{}\_)$ is defined by $(A\hat{}\_)(X) = \{\, A \xrightarrow{f} X \mid f$ is a function from A to $B\,\}$. The* ***adjunction pattern*** *is a bijective correspondence that may be written alternatively as*

$$\mathcal{S}((A \times \_)(B), C) \leftrightarrow \mathcal{S}(B, (A\hat{}\_)(C)) \quad \text{or}$$
$$\mathcal{S}(A \times B, C) \leftrightarrow \mathcal{S}(B, C^A) \quad \text{or}$$
$$C^{A \times B} \leftrightarrow (C^A)^B$$

or as

$$\frac{A \times B \longrightarrow C}{\overline{A \longrightarrow C^A}}$$

in which $\mathcal{S}(X,Y) = Y^X$ *is the set of functions from* X *to* Y*. The moving parts are* A, B *and* C*.*

Example 3. *(Category Theory) The* ***adjunction pattern*** *between the Cartesian product operator and the exponentiation operator in the previous example is actually just a corner of a large expanse of instances of the* ***adjunction pattern****, as in category theory [30, page 85], which are abbreviated as*

$$\frac{F(B) \to C}{\overline{B \to G(C)}}$$

and which may be read as saying "arrows from $F(B)$ *to* C *correspond to arrows from* B *to* $G(C)$*, and vice versa."*

The reader is trusted to discern a rough formal analogy between these examples from rather remote disciplines (statistics [9, page 154] and category theory [24, page 12].

Example 4. *(Exterior Algebra) The generalized theorem of Stokes:*

Theorem 1. *If* ω *is an* n*-form on a manifold* $\mathbf{M}$ *and* $\mathbf{p}$ *is an* $(n+1)$*-chain, then*

$$\int_{\partial \mathbf{p}} \omega = \int_{\mathbf{p}} d\omega\,.$$

Proof. [15]. □

This match to the ***adjunction pattern*** *may be re-presented as The theorems of Green and Gauss are special cases of the generalized theorem of Stokes.*

Example 5. *(Linear Algebra)*

Theorem 2. *If* $A \xrightarrow{T} B$ *is a linear transformation between two finite-dimensional inner product spaces* A *and* B*, then there exists a unique linear transformation* $B \xrightarrow{T^\dagger} A$ *such that*

$$< T^\dagger y, x > = < y, Tx >$$

for all vectors $x \in A$ *and* $y \in B$*.*

Proof. [29, page 396]. □

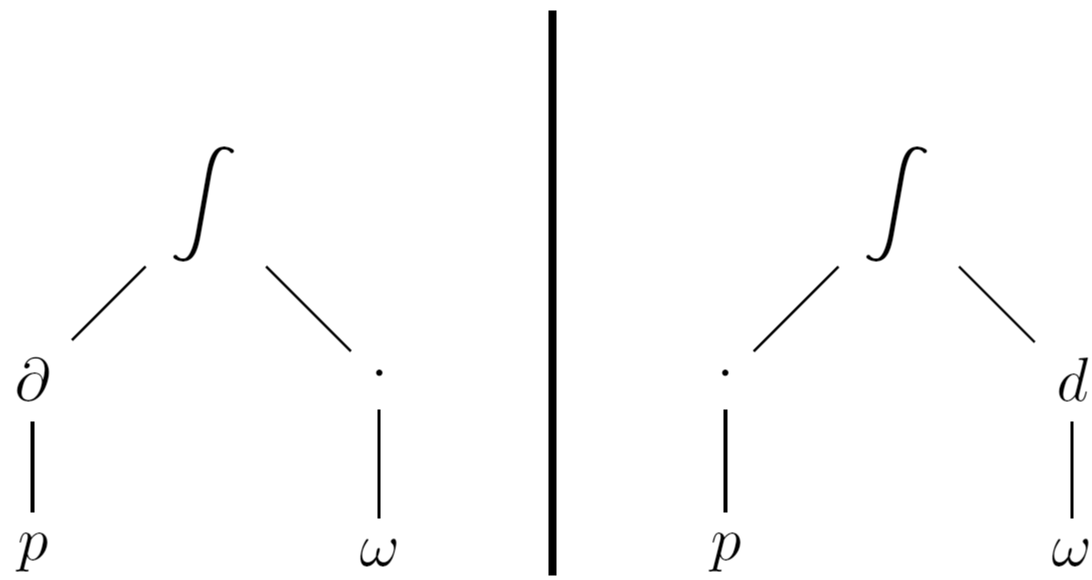

Figure 4: Stokes' Theorem.

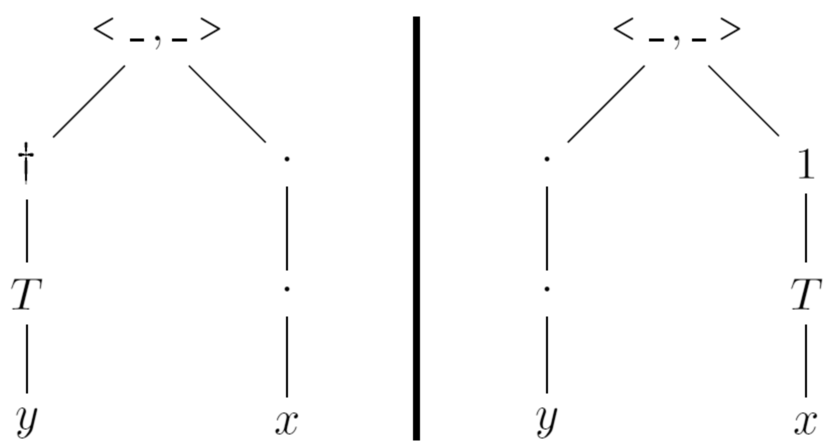

Figure 5: Adjoint linear transformation.

Re-presented:

*The linear transformation $T^\dagger$ is called the **adjoint transformation** of T. The concept of adjoint transformation is utterly crucial in quantum mechanics.*

*For example, if $A \xrightarrow{T} B$ and $T^\dagger = T$ (i.e., T is **self-adjoint**), then T is called an **observable** [55].*

*In the 1950s Daniel Kan, working in algebraic topology, discovered "what is undoubtedly the fundamental notion of category theory" [21, pages 294–329][34, page 109]. It was like the discovery of "black gold," changing the world forever – well, at least the mathematical world. What Samuel Eilenberg and Saunders Mac Lane did for "naturality" in mathematics [12, pages 231–294], is what Daniel Kan did for "universality." Samuel Eilenberg first perceived the substantial notation analogy between adjoint linear transformations and Kan's new concept about functors between categories. That is why they are called adjoint functors. Even though linear transformations are not functors between categories, the pattern shared with adjoint functors takes primary position in the history of **adjunction patterns**.*

Example 6. *(Categorical Logic) Sometimes K-12 students are introduced to representations of propositional logic relations using "Venn diagrams." Closed curves (often, circles) in the plane represent predicates. That is to say, interior points of a closed curve represent elements of the "universe" which satisfy a predicate associated with the curve. The "universe" is a rectangle, say, that encloses all the curves in consideration. In this representation, unions of interiors correspond to disjunctions of predicates, and intersections of interiors correspond to conjunctions of predicates. The exterior of a curve corresponds to the negation of the corresponding predicate. (Note that this pictorial representation is justified by the Jordan-Veblen Curve Theorem.) It is possible to extend pictorial representations to include the universal and existential quantifiers of first-order logic, as in Fig. (6).*

In a multi-sorted first-order language, if $S(x,y)$ is a binary predicate formula, then its semantic interpretation is a subset of a Cartesian product $X \times Y$ where X and Y are sets in which variables x and y have their values. The Cartesian product has projection maps, including $X \times Y \xrightarrow{\pi_X} X$. In general, any subset $A \subseteq X \times Y$ may be considered the semantic interpretation of an "abstract binary predicate," and likewise, any subset $T \subseteq X$ is an abstract unary-predicate over X. The projection map π_X induces a map (by abuse of notation also called π_X) $\mathcal{P}(X \times Y) \xrightarrow{\pi_X} \mathcal{P}(X)$, defined by

$$\pi_X(A) \coloneqq \{\, x \in X \mid (x,y) \in X \times Y \,\}\,.$$

Since this map preserves the subset relation in the sense that $U \subseteq V \subseteq X \times Y$ implies $\pi_X(U) \subseteq \pi_X(V) \subseteq X$, it may be thought of as a functor from the partial-order category of subsets of $\mathcal{P}(X \times Y)$ to the partial-order category of $\mathcal{P}(X)$. F. W. Lawvere's seminal observation is that this functor has both a left adjoint (existential reification) and a right adjoint (universal reification) functor. This is obvious by consideration of Fig. (6). At (Left) assume the outer rectangle

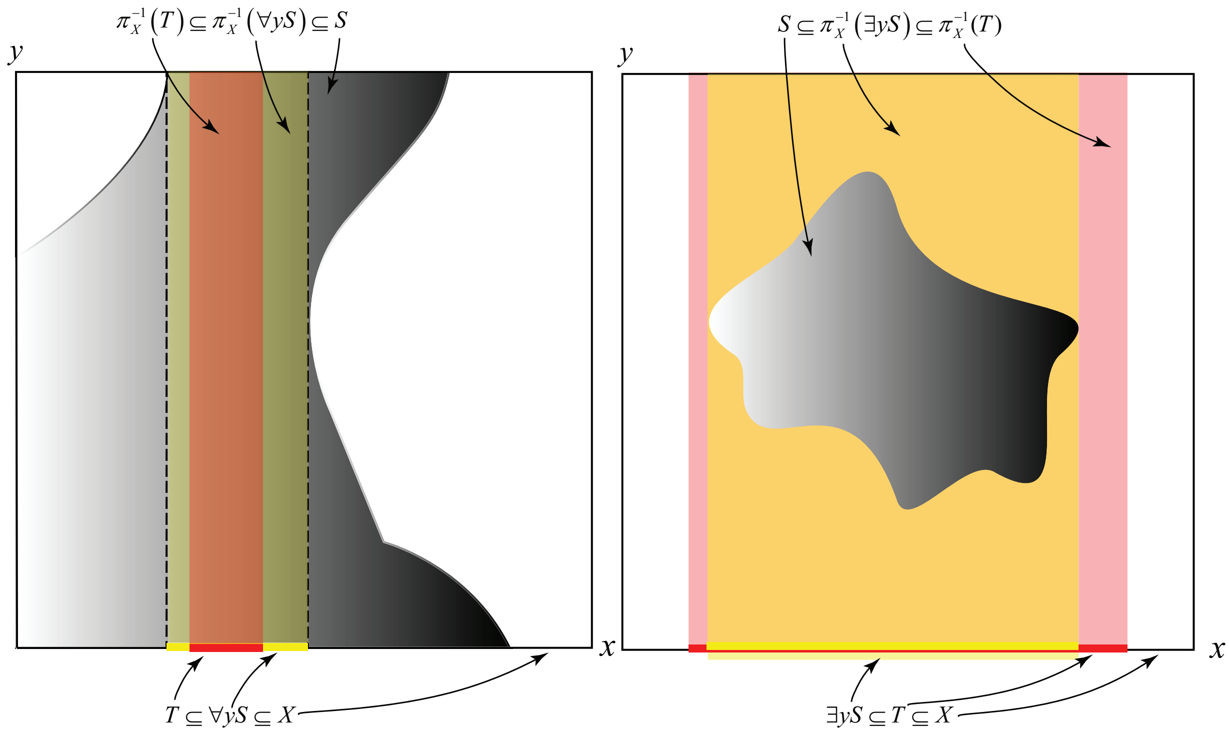

Figure 6: (Left) Extent of universal quantifier. (Right) Extent of existential quantifier.

represents $X \times Y$ with values x of X on the horizontal axis, and likewise those of y on the vertical axis. Then the binary predicate S may be represented by the greyish subset. The universal quantification predicate formula $(\forall y)S(x, y)$ binds y leaving x free, resulting in a unary predicate formula over X.

By the set-theoretic Axiom of Separation, If X is a set and $\Phi(X)$ is a first-order formula with one free variable, x, then $\{\, x \in X \mid \Phi(x) \,\} \subseteq X$, and it may be called a ***formulable set****, as opposed to just some unspecified but named abstract subset of X.*

This formula $(\forall y)S(x, y)$ corresponds to the formulable set $\forall y S := \{\, x \in X \mid (\forall y)S(x, y) \,\}$. If $T \subseteq \forall y S$, then application of the functor π_X yields $\pi_X^{-1}(T) \subseteq \pi_X \forall y S$, but the latter is of course a subset of S. The result is a natural bijection between inclusions $\pi_X^{-1}(T) \subseteq S$ and inclusions $T \subseteq \forall y S$. That is, there is a match to the ***adjunction pattern*** *for universal quantification,*

$$\frac{\pi_X^{-1}(T) \subseteq S}{\overline{T \subseteq \forall y S}}$$

An entirely symmetrical argument based on (Right) yields the ***adjunction pattern*** *for existential quantification,*

$$\frac{\underline{\exists y S \subseteq T}}{S \subseteq \pi_X^{-1}(T)}$$

> *The idea that logical operations, all logical operations, should appear as adjoints to basic functors was one of Lawvere's convictions and motivation... Lawvere pursued his work in the categorification of logic and presented two papers at the Meeting of the Association of Symbolic Logic, one in 1965 and the other in 1966. ... It is entirely clear that Lawvere is trying to extend his results obtained for algebraic theories to first-order theories in general.... [I]t is the first time in print that the existential quantifier is presented as an adjoint [35].*

4 The Recursion Pattern

Example 7. *(Linguistics) Hard-core cognitive capacities are central to being human. Certainly among them is the capacity for recognition of hierarchically nested patterns [42]. In a word,* ***recursion****. It is evident in language, music, dance, sport, and in just ordinary daily activities ("Hold on, there is someone at the door,..."). Whether it is "analytic recursion"*

$$(((((\cdots \bullet \cdots)))))$$

(in the sense of starting at the top and working down to the bottom (as in, say, defining a business organization with a hierarchical chart), or it is "synthetic recursion"

$$\cdots(((((\bullet)))))\cdots$$

(as in generating the grammatical expressions of (natural or artificial)(verbal or diagrammatic) language), no other species plumbs the depths, or achieves the heights, as do human beings with recursion.

Perhaps the concept recursion has had no greater impact on a scientific discipline than on linguistics. If a grammar for a natural language must be formalized by a finite number of rules, yet account for the grammaticality of an unlimited number of possible sentences, then recursion is the ticket.

In the epochal work [6], Noam Chomksy emphasized that "In general, the assumption that languages are infinite is made for the purpose of simplifying *the description. If a grammar has no recursive steps … it will be prohibitively complex – it will, in fact, turn out to be little better than a list of strings or of morpheme class sequences in the case of natural language. If it does have recursive devices, it will produce infinitely many sentences." In that paper (3600 citations) he proves a theorem that describes the increasing generative capacity of three abstract models of language. A modern take on this so-called "Chomsky Hierarchy" applies elementary category theory, in what may be called "coalgebraic abstract linguistics." The simplest level of the Chomsky Hierarchy is called* ***regular language****. Details of this are described below. The next level is called* ***context-free language****, and a coalgebraic version is provided by [50]. The top-level* ***phrase-structure language*** *is discussed coalgebraically in [18]. The higher levels deploy increasingly sophisticated category theory, but not beyond what would be covered in an advanced undergraduate course in computer science.*

> *For example, a sentence such as "He assumed that now that her bulbs had burned out, he could shine and achieve the celebrity he had always longed for." Tucked inside the one thought beginning "He assumed" are four more thoughts, tucked inside one another: "Her bulbs had burned out," "He could shine," "He could achieve celebrity," and "He had always longed for celebrity." So five thoughts, starting with "He assumed," are folded and subfolded inside twenty-two words . . . recursion . . . On the face of it, the discovery of recursion was a historic achievement. Every language depended upon recursion – every language. Recursion was the one capability that distinguished human thought from all other forms of cognition . . . recursion accounted for man's dominance among all the animals on the globe [51].*

A simple example of recursive grammar may be formalized using re-write rules:[1]

$$\text{S} \to \text{NP-VP} \tag{2}$$
$$\text{VP} \to \text{V-NP} \tag{3}$$
$$\text{VP} \to \text{V-S} \tag{4}$$
$$\text{NP} \to \text{N} \tag{5}$$

In words, Equ.(2) says, "A sentence S is a noun phrase NP followed by a verb phrase VP. Equ.(3) says a verb phrase is a verb V followed by a noun phrase. Equ.(4) is where *recursion* rises, since it says a verb phrase may itself be a sentence. Then Equ.(5) says a noun phrase is a noun N. Thus, sentences

[1] also known as "production rules," or "replacement rules."

may be nested any number of times, resulting in an unlimited number of possible grammatical sentences [41].

Any scientific hypothesis must answer to the facts on the ground. A genuine scientific adventurer – like the fictional Indiana Jones – Daniel L. Everett lived at length with his family very deep in Brazil's Amazon basin among a remarkable people, the Pirahã (pronounced PEE-DA-HANNH). The fact is that the language of this tribe *has no recursion*:

> Pirahã, had no recursion, none at all, immediately reducing Chomsky's *law* to just another feature found in most languages ... it was the Pirahã's own distinctive culture, their unique ways of living, that shaped the language – not any "language organ," not any "universal grammar" or "deep structure" or "language acquisition device" that Chomsky said all languages had in common [51].

That scientific discovery engendered considerable controversy among linguists. Worse, there is disagreement about the meaning of "*recursion*[48]":

> As will be shown in detail, the notion of 'recursion' was fundamentally ambiguous when it began to be used by linguists in the 1950s, and (as the subtitle of this article implies), these ambiguities have persisted to the present day. This unfortunate (and needless) perpetuation of imprecision has had a deleterious impact upon recent discussions of the role of recursion in linguistic theory [27].

Definition 1. If A is a non-empty set, then A^* denotes the set of finite sequences $\overrightarrow{a} = (a_1, \ldots, a_n)$ of length $0 \leq n \in \mathbb{N}$, including the unique sequence $\varepsilon = ()$ of length 0. A sequence (a) of length 1 is identified with the element $a \in A$, so $A \subset A^*$. Elements of A are called **symbols** and elements of A^* are called **strings**. A set $L \subset A^*$ of strings is called an **abstract language**. An abstract language for which there is a method to decide whether a string is an element of it is called a **language**.

Definition 2. If $\mathcal{S} \xrightarrow{F} \mathcal{S}$ is an endofunctor of the category $\mathcal{S}$ of sets, then a map $X \xrightarrow{\alpha} FX$ is called a **coalgebra** of F, and if $Y \xrightarrow{\beta} FY$ is a coalgebra of F, then a map $X \xrightarrow{f} Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & FX \arrow[d, "Ff"] \\ Y \arrow[r, "\beta"'] & FY \end{tikzcd}$$

is commutative (i.e., if $Ff \circ \alpha = \beta \circ f$) is a **coalgebra map** from α to β. The coalgebras of F and their maps form the category of F-coalgebras, which is denoted by $\mathbf{Coalg}(F)$.

Definition 3. *The set* $\{0,1\}$ *is denoted by* Ω. *For any set* X *the subsets* $X_0 \subseteq X$ *correspond exactly to the functions* $X \xrightarrow{\chi} \Omega$ *such that* $\chi(x_0) = 1 \,\mathrm{if}\, \mathrm{x}_0 \in \mathrm{X}_0 \,\mathrm{else}\, 0$. *Thus,* $Z := \Omega^{A^*}$ *may denote the set of abstract languages.*

Definition 4. *For a set* A *of symbols a coalgebra of the endofunctor* $\mathcal{S} \xrightarrow{D} \mathcal{S}$ *defined by*

$$DX := \Omega \times X^A \tag{6}$$

is called an ***automaton*** *(finite automaton if* X *is finite). If* $X \xrightarrow{(\phi,\alpha)} \Omega \times X^A$ *is an automaton, then an element* $x_0 \in X$ *is called a* ***state*** *of* (ϕ,α)*, and if* $\phi(x) = 1$ *then* x *is an* ***acceptance state****.*

If $x, y \in X$ *and* $a \in A$ *satisfy* $\alpha(x)(a) = y$*, then* a ***induces a transition*** *from* x *to* y*, and that circumstance is denoted by* $x \xrightarrow{a} y$.[2] *If a sequence of transitions*

$$x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \rightarrow \cdots \xrightarrow{a_n} x_n \tag{7}$$

ends at an acceptance state, i.e., if $\chi(x_n) = 1$*, then then string* $\vec{a} := (a_1, a_2, \ldots, a_n)$ *from* x_0 *to* x_n *is* ***accepted*** *by the automaton* (ϕ,α)*. The (possibly empty) set* $L(x_0) \subseteq A^*$ *of strings* $\vec{a}$ *from* x_0 *to* some *acceptance state is called the* ***language at*** x_0*, hence there exists a map* $X \xrightarrow{L} Z$*. In short, a language*

$$L(x_0) := \{\, \vec{a} \in A^* \mid \textit{there exist transitions}\, Equ.(7) \textit{ such that } \vec{a} = (a_1, \ldots, z_n) \textit{ and } \phi(x_n) = 1) \,\}\,. \tag{8}$$

is a formulable set.

Definition 5. *The* D*-coalgebra*

$$Z \xrightarrow{(\chi,\zeta)} \Omega \times Z^A \tag{9}$$

$$\chi(L) := 1 \,\mathit{if}\, \varepsilon \in L \,\mathit{else}\, 0 \tag{10}$$

$$\zeta(L)(a) := \{\, \vec{a} \in A^* \mid a\vec{a} \in L \,\} \tag{11}$$

is called the ***abstract language coalgebra****.*

[2] This arrow notation is *not* intended to suggest a map in some category. It is best to consider it a symbol-labelled arrow in a directed graph whose tail and head are states.

Theorem 3. If

$$L_0 \xrightarrow{a_1} L_1 \to \cdots \to L_{n-1} \xrightarrow{a_n} L_n$$

is a sequence of transitions from L_0 to L_n in (χ, ζ), then $\vec{a} = (a_1, \ldots, a_n) \in L_0$ if and only if $\vec{a}$ is accepted by (χ, ζ).

Proof. By "downward induction,"

$$\begin{aligned}
\vec{a} \text{ is accepted by } (\chi,\zeta) &\Leftrightarrow \chi(L_n) = 1 \\
&\Leftrightarrow \varepsilon \in L_n \\
&\Leftrightarrow \varepsilon \in \{\, \vec{x} \in A^* \mid a_n \vec{x} \in L_{n-1} \,\} \\
&\Leftrightarrow a_n \varepsilon \in L_{n-1} \\
&\Leftrightarrow a_n \in L_{n-1} \\
&\Leftrightarrow a_n \in \{\, \vec{x} \in A^* \mid a_{n-1} \vec{x} \in L_{n-2} \,\} \\
&\Leftrightarrow a_{n-1} a_n \in L_{n-2} \\
&\vdots \\
&\Leftrightarrow a_1 \cdots a_n \in L_0 \,.
\end{aligned}$$

□

Theorem 4. $\chi, \zeta)$ is a terminal object in $\mathbf{Coalg}(D)$. (It is also called the "final coalgebra" for D).

Proof. It must be demonstrated that for any coalgebra $X \xrightarrow{(\phi,\alpha)} \Omega \times X^A$ there exists a unique map $X \xrightarrow{f} Z$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "(\phi,\alpha)"] \arrow[d, "f"'] & \Omega \times X^A \arrow[d, "1 \times f^A"] \\
Z \arrow[r, "(\chi,\zeta)"'] & \Omega \times Z^A
\end{tikzcd} \tag{12}$$

is commutative. Diagram (12) is equivalent to equations

$$\phi(x) = \chi(f(x)) \tag{13}$$
$$f(\alpha(x)(a)) = \zeta(f(x))(a) \tag{14}$$

for all $x \in X$. If $\vec{a} = (a_1, \ldots, a_n) \in A^*$ and Equ. (7) is a sequence of transitions from x_0 to x_n in (ϕ, α), then $\alpha(x_i)(a_{i+1}) = x_{i+1}$ for $0 \le i < n$, so by Equ. (14) $\zeta(f(x_i))(a_i) = f(\alpha(x_i)(a_{i+1})) = f(x_{i+1})$ for $0 \le i < n$. Therefore

$$f(x_0) \xrightarrow{a_1} f(x_1) \to \cdots \to f(x_{n-1} \xrightarrow{a_n} f(x_n)$$

is a sequence of transitions from $f(x_0)$ to $f(x_n)$ in (χ,ζ). Hence,

$$\begin{aligned}\overrightarrow{a} = (a_1,\ldots,a_n) \text{ is accepted by } (\chi,\zeta) &\Leftrightarrow \chi(f(x_n)) = 1\\ &\Leftrightarrow \phi(x_n) = 1 \text{ by } (13)\\ &\Leftrightarrow \overrightarrow{a} \in L(x_0) \text{ by Theorem } (3)\,.\end{aligned}$$

Hence f can and must be defined by $f = L$. □

Theorem 5. If an endofunctor has a final coalgebra ζ, then ζ is an isomorphism.

Proof. In Diag.(15), the upper square obviously commutes. Since $X \xrightarrow{\zeta} EZ$ is a final coalgebra by assumption, and since $EZ \xrightarrow{E\zeta} EEZ$ is a coalgebra, there exists a unique morphism $EZ \xrightarrow{\phi} Z$ such that the bottom square is commutative. Since the identity morphism $Z \xrightarrow{1_Z} Z$ renders the outer rectangle commutative, it follows that $\phi \circ \zeta = 1_Z$ and $\zeta \circ \phi = E\phi \circ E\zeta = 1_{EZ}$. Therefore, $Z \cong E$.

$$\begin{tikzcd} Z \arrow[r, "\zeta"] \arrow[d, "\zeta"'] & EZ \arrow[d, "E\zeta"] \\ EZ \arrow[r, "E\zeta"] \arrow[d, dotted, "\phi"'] & EEZ \arrow[d, dotted, "E\phi"] \\ Z \arrow[r, "\zeta"'] & EZ \end{tikzcd} \tag{15}$$

□

Consequently, $Z \cong \Omega \times Z^A = E_A(Z)$, which conforms to the **recursion pattern**. Moreover, (χ,ζ) is (up to isomorphism) the final coalgebra for $\mathbf{Coalg}(D)$.

5 Biological Recursion Patterns

Example 8. *(Biology) The "Central Dogma of Molecular Biology" was introduced by Francis Crick in 1958 [5, 10, 17]. It is a statement about the transfer of information encoded in the molecular structures of three basic kinds: DNA, RNA, and proteins. The "general transfers" occur in all cells:*

$$\begin{aligned}&\text{DNA} \to \text{DNA}\\ &\text{DNA} \to \text{RNA}\\ &\text{RNA} \to \text{Protein}\end{aligned}$$

"The flow of genetic information from DNA to RNA (transcription) and from RNA to protein (translation) occurs in all living cells [1]."

The "special transfers" may occur in some circumstances:

$$\begin{aligned}&\mathrm{RNA} \to \mathrm{RNA}\\&\mathrm{RNA} \to \mathrm{DNA}\\&\mathrm{DNA} \to \mathrm{Protein}\end{aligned}$$

And there are information transfers not known to occur:

$$\begin{aligned}&\mathrm{Protein} \to \mathrm{Protein}\\&\mathrm{Protein} \to \mathrm{DNA}\\&\mathrm{Protein} \to \mathrm{RNA}\end{aligned}$$

Crick concludes [10] by firmly stating that "the discovery of just one type of present day cell which could carry out any of the three unknown transfers would shake the whole intellectual basis of molecular biology." The "principle of recursion of genomic function" published by Andras J. Pellionisz [40] asserts precisely that the unknown transfer

$$\mathrm{Protein} \to \mathrm{DNA} \tag{16}$$

does indeed occur, specifically during the development of Purkinje cells. He defines recursion "as a process of defining functions in which the function being defined is applied within its own definition," and he appears to conflate "recursion" with "feedback," as in "these recursive feedback processes then snowball into evolving (protein) structures, governed by DNA." However alluring, his views seem not to have rattled the foundations of molecular biology. Then again, see [39].

Example 9. *(Theoretical Biology) Theoretical biologists are searching for a mathematical theory in terms of which to articulate a "theory of organism" [46]. The abstract of an article [36] includes the sentence, "We interpret biological individuation as a second order one, i.e. as a recursive procedure through which physical individuation is also acting on "its own theatre"." This could stand thorough explanation, especially as in the article there also appears the* ***recursion pattern****:*

> *[A]n autopoietic system like a cell is also the result of its own interactions with its environment so that its organizational unity is preserved, as a fixed point of a given equation:*
>
> $$F = \phi F\,.$$

It would probably take a lot of work to figure out the meaning behind such hints.

6 Mathematical Recursion Patterns

Example 10. *(Non-Wellfounded Set) In conventional axiomatic set theory (such as that of Zermelo-Frankel) the equation $X = \{X\}$, which conforms to the **recursion pattern**, has no solution. That is because if it did have a solution, say X, then there would be an infinitely descending sequence of membership relations,*

$$\cdots \in X \in X \in X \in X\,,$$

but that is precluded by the Axiom of Regularity (a.k.a., "Well-Foundedness Axiom"). The fecund study of set theory without the Well-Foundedness Axiom has interesting mathematical consequences, and useful applications to computer science [37].

Example 11. *(Brouwer's Fixed Point Theorem) The following theorem is a well-known fundamental result in algebraic topology.*

Theorem 6. *If X denotes the closed unit ball in a finite-dimensional Euclidean space, then for any continuous endomap $X \xrightarrow{f} X$, there exists $x \in X$ such that $x = f(x)$, which conforms to the **recursion pattern**.*

One consequence of this is the celebrated Jordan-Veblen Curve Theorem, without which, for example, Peirce's "existential graphs" would not exist:

> *To indicate negation in his original version of EGs, Peirce used an unshaded oval enclosure, which he called a cut or sometimes a sep because it separated the sheet of assertion into a positive (outer) area and a negative (inner) area [47].*

Example 12. *(The Exponential Function)*

Theorem 7. *If $X \subset \mathbb{R}^{\mathbb{R}}$ denotes the set of differentiable endomaps of the real numbers, and*

$$X \xrightarrow{\frac{d}{dx}} X$$

denotes the differentiation endomap, then there exists a unique function $e \in X$ such that $e(0) = 1$ and

$$e = \frac{d}{dx} e\,.$$

*which conforms to the **recursion pattern**.*

Example 13. *(Primitive Recursion) There is proof by induction, and there is definition by recursion. Both are handholds on infinity. In the case of induction, a predicate $P(x)$ about natural numbers $x \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is proved in two steps. First, the induction is initiated by proving that $P(0)$. This is like lighting a fuse. Second, it is demonstrated that if $P(n)$, then $P(n+1)$ has a proof. Thus, there is an infinite chain of proofs for $P(0), P(1), P(2), P(3), \ldots$. The fuse burns.*

The most primitive notion of recursion is to define a sequence given a function $X \xrightarrow{E} X$ and an initial element $x_0 \in X$. The sequence begins with x_0, the next element is $x_1 = E(x_0)$, the next element is $x_2 = E(x_1) = E(E(x_0))$, and so on: $x_{n+1} = E(X_n)$. The result is the infinite sequence (possibly with repetitions!) $x_0, x_1, x_2, x_3, \ldots$. This notion of primitive recursion is adroitly turned around by F. W. Lawvere to characterize the natural numbers in what is called the "Peano-Lawvere Axiom" [35]:

Definition 6. A **natural numbers object** is a structure with data

$$1 \xrightarrow{z} \mathbb{N} \xrightarrow{s} \mathbb{N}$$

which satisfies the condition that if

$$1 \xrightarrow{x_0} X \xrightarrow{E} X$$

then there exists a unique function $\mathbb{N} \xrightarrow{x} X$ such that the diagrams

$$\begin{tikzcd}
 & \mathbb{N} \arrow[d, "x"] & \mathbb{N} \arrow[r, "z"] \arrow[d, "x"'] & \mathbb{N} \arrow[d, "x"] \\
1 \arrow[ur, "z"] \arrow[dr, "x_0"'] & & & \\
 & X & X \arrow[r, "E"'] & X
\end{tikzcd}$$

are commutative, i.e., $x \circ z = x_0$ and $x(s(n)) = E(x(n))$. So, z should be interpreted specifically as 0, and s as the successor function $s(n) = n + 1$.

Example 14. *(Attractors in Dynamical Systems) The seemingly simple type of structure $X \xrightarrow{E} X$ appears in one guise or another throughout mathematics, physics, and computer science. Often it is a **dynamical system** with **states** X and **law of evolution** E. So, given an initial state $x_0 \in X$, the Peano-Lawvere Axiom returns with the evolution of x_0 through a sequence of subsequent states [26, page 137].*

Recall that if X is a set, then another set exists whose elements are the subsets of X. The set of subsets of X is denoted by $\mathcal{P}X$. If $X \xrightarrow{f} Y$ is a *function, then it induces a function* $\mathcal{P}X \xrightarrow{\mathcal{P}f} \mathcal{P}Y$ *defined by* $(\mathcal{P}f)(A) \coloneqq f(A) = \{\, f(x) \in Y \mid x \in X \,\}$*, namely the image of* A *by the function* f*.*

Theorem 8. If X is a non-empty finite set and $X \xrightarrow{E} X$, then there exists a subset $A \subseteq X$ such that

$$A = (\mathcal{P}f)(A)\,,$$

which conforms to the **recursion pattern**.

Proof. Since X is non-empty, there exists an element $x_0 \in X$. Since X is finite, the Peano-Lawvere sequence $x_0, x_1 = E(x_0), x_2 = E(x_1), \ldots, x_{n+1} = E(x_n), \ldots$ is finitely valued. Hence, there exist $m, n \in \mathbb{N}$ such that $x_m = x_n$. So, there exists a least n such that $x_m, \ldots, x_{n-1}$ are distinct and $x_n = x_m$. Define $A = \{\, x_m, \ldots, x_{n-1} \,\}$. Then

$$\begin{aligned} E(A) &= E\{\, x_m, \ldots, x_{n-1} \,\} \\ &= \{\, E(x_m), \ldots, E(x_{n-1}) \,\}(\textbf{distribution pattern}) \\ &= \{\, x_{m+1}, \ldots, x_n \,\} \\ &= A\,. \end{aligned}$$

□

That is to say, A is a fixed point of the induced dynamical system. Note that although an attractor is a fixed point of the induced dynamical system, its elements are in constant (cyclic) motion in the underlying dynamical system. This is akin to the retention of a human being's identity even though her cells continually die and are replaced.

A "one-dimensional cellular automaton" (CA) is a finite dynamical system in which the states are finite rows of "cells" colored black or white, and the law of evolution is defined by stipulating how a state changes, usually in terms of the colors of neighboring cells.

> *State space (also called phase space) is the set of all possible CA global states. In a finite CA, state space is finite; thus, any trajectory must eventually encounter a repeat of a global state that occurred at an earlier time. Because the system is deterministic, the trajectory will become trapped in this repeating sequence of states, a cyclic attractor, with a specific period of 1 or more.*
>
> *States are either part of the attractor or belong to a transient, a sequence of states leading to the attractor. If transients exist, there must be states at their extremities (garden-of-Eden states), unreachable by evolution from any other state. The set of all possible transients leading to an attractor, plus the attractor itself, is the basin*

> *of attraction of that attractor. State space is populated by one or more basins of attraction. These basins of attraction constitute the dynamical flow imposed on state space by the CA transition function.*
>
> *A portrait of this global behaviour is the basin of attraction field, a discrete analogue of the familiar basin of attraction field found in the phase space of a continuous dynamical system, known as the system's phase portrait [53].*

Cataloging visual presentations of the phase portraits of finite dynamical systems is a favorite sport of many computer programmers. The book [53] is one example; "A New Kind of Science" by Stephen Wolfram [52] is another.

Example 15. *(Lawvere Fixed-Point Theorem [3])*

Theorem 9. If there exists a surjection $A \xrightarrow{e} B^A$ and an endomap $B \xrightarrow{f} B$, then there exists $b \in B$ such that

$$b = f(b)\,,$$

which exemplifies the **recursion pattern**.

Proof. The composition $A \xrightarrow{(1_A,e)} A \times B^A \xrightarrow{\varepsilon_B^A} B \xrightarrow{f} B$ is a map from A to B, hence there is $a \in A$ such that $e(a) = f \circ \varepsilon_B^A \circ (1_A, e)$. Calculate

$$\begin{aligned} e(a)(a) &= (f \circ \varepsilon_B^A \circ (1_A, e))(a) \\ &= (f \circ \varepsilon_B^A)(a, e(a)) \\ &= f(\varepsilon_B^A(a, e(a))) \\ &= f(e(a)(a))\,. \end{aligned}$$

□

Example 16. *(Spectral Theorem) The* ***adjunction pattern*** *and the* ***recursion pattern*** *intersect in the spectral theorem of linear algebra. Assume that U is an inner product (x, y) space over the real numbers $\mathbb{R}$. A linear endomap $U \xrightarrow{T} U$ is* ***self-adjoint*** *if*

$$(Ta, b) = (a, Tb)$$

for all vectors $a, b \in U$, which is a case of the ***adjunction pattern****. An* ***eigenvector*** *of T is a vector $a \in U$ such that $T(a) = \lambda a$ for some real number $\lambda \in \mathbb{R}$, which is equivalent to*

$$a = \frac{1}{\lambda} T(a)\,,$$

thereby conforming to the ***recursion pattern****. The number λ is called an* ***eigenvalue*** *of T. Any scalar multiple of an eigenvector is also an eigenvector* *with the same eigenvalue, and the sum of any two eigenvectors with the same eigenvalue is also an eigenvector of it. Hence, the eigenvectors of an eigenvalue λ form a linear subspace of U, which may be denoted by $U(\lambda) \subset U$ and called the* ***eigenspace*** *of λ. The set of all eigenvalues of T is called the* ***spectrum*** *of T.*

Theorem 10. *(Spectral Theorem [29])*If U is finite-dimensional and $\{ \lambda_1, \ldots, \lambda_n \}$ is the spectrum of a self-adjoint linear endomap $U \xrightarrow{T} U$, then

1. U is the direct sum of the eigenspaces of T, i.e., $U = U(\lambda_1) \oplus \cdots \oplus U(\lambda_n)$;
2. the eigenspaces $U(\lambda_1), \cdots, U(\lambda_n)$ are mutually orthogonal;
3. the dimension of $U(\lambda_i)$ is equal to the "multiplicity" of λ.

Example 17. *(Computability) Computers can crash. In theoretical computer science, this fact is formalized by the idea that computable functions – whatever these are – are not necessarily totally-defined, that is to say, computable functions are partially-defined.*

Definition 7. A **totally-defined function** F from a set X to a set Y is a structure with data $F \subseteq X \times Y$ satisfying condition that for every $x \in X$ there exists exactly one $y \in Y$ such that $(x, y) \in F$. This structure is denoted by

$$X \xrightarrow{F} Y$$

and $(x, y) \in F$ is re-presented as an equation $y = F(x)$. A **partially-defined function** F from X to Y is a single-valued relation $F \subseteq X \times Y$, which means that if $(x, y_1) \in F$ and $(x, y_2) \in F$, then $y_1 = y_2$. (It is possible, accordingly, that F is completely empty, in which case F is a **totally-undefined function**.)

Theorem 11. If $\perp$ is not an element of Y, then a partially-defined function F from X to Y corresponds exactly to a totally-defined function $X \xrightarrow{F} Y_\perp$, where $Y_\perp$ is defined to be the disjoint union of Y with the singleton set $\{ \perp \}$.

Proof. $F^{-1}(\perp) = \{ x \in X \mid \text{there is no } y \in Y \text{ such that } (x, y) \in F \}$. □

Henceforth, a partially-defined function is presented using the $\perp$ notation. That said, computability theory of the natural numbers $\mathbb{N}$ is about certain partially-defined functions

$$\overbrace{\mathbb{N} \times \cdots \times \mathbb{N}}^{n} \xrightarrow{f} \mathbb{N}_\perp .$$

and the set of all partially-defined numerical functions is

$$\mathbb{N}_\perp^{\overbrace{\mathbb{N} \times \cdots \times \mathbb{N}}^{n}} .$$

The informal notion of computability can be formalized in the language of the theory of sets and functions in several ways. There are theorems, however, *proving that each of these seemingly disparate formalizations defines exactly the same set of partially-defined functions. The Church-Turing Thesis is the assertion that any proposed formalization of computability will lead to that same set [43, page 20]. Thus, a theory of computation is founded on the language of the theory of sets, which in turn is based on the notion of element-hood. Nicely enough, there are some basic theorems in that theory from which many other important theorems can be derived, necessarily still founded on the theory of sets. In other words, basic theorems may be taken as axioms for a more abstract theory of computation.*

F. W. Lawvere suggested radically that mathematics could be done without the notion of element-hood.

> *[T]he substance of mathematics resides not in Substance (as it is made to seem when [element membership] is the irreducible predicate, with the accompanying necessity of defining all concepts in terms of a rigid element-hood relation) but in Form (as is clear when the guiding notion is isomorphism-invariant structure, as defined, for example, by universal mapping properties) [25].*

In particular, the theory of computation may "proceed virtually without the use of elements" [7, 38]. This meditation does not ascend to that level of abstraction, but the axioms and theorems discussed below all have their counterparts in the theory of computation without elements.

6.1 Axioms

[Axiom E] is that for any natural number $n \in \mathbb{N}$ there exists a totally-defined function ϕ^n whose value $\phi^n(x)$ is a function with n inputs, i.e., for $y \in \overbrace{\mathbb{N} \times \cdots \times \mathbb{N}}^{n}$ the value $\phi^n(x)(y)$ is either a natural number, or is undefined, $\phi^n(x)(y) \in \mathbb{N}_\perp$:

$$\frac{n \in \mathbb{N}}{\mathbb{N} \xrightarrow{\phi^n} PC_n \subset \mathbb{N}_\perp^{\overbrace{\mathbb{N} \times \cdots \times \mathbb{N}}^{n}}}$$

[Axiom U] ("The Universality Theorem") declares that there exists a natural number u (for "universal") such that the 2-input function $\phi^2(u)$ is a totally-defined computable function that "simulates" any partially-defined computable function, in the sense that for any partially-defined computable function $\phi^1(p)$, its value $\phi^(p)(x)$ is calculated by $\phi^2(u)(p,x)$ at x:

$$\frac{}{u \in \mathbb{N} \qquad \dfrac{p, x \in \mathbb{N}}{\phi^1(p)(x) = \phi^2(u)(p,x)}}$$

[Axiom P] ("The Parametrization Theorem" a.k.a. "The S-M-N Theorem") stipulates that for any natural numbers $m > 0$ and $n > 0$ there exists a totally-defined computable function with $n+1$-inputs such that a kind of **association pattern**:

$$\dfrac{m, n \in \mathbb{N}}{s_n^m \in TC_{m+1} \quad \dfrac{p \in \mathbb{N} \quad \overrightarrow{x} \in \mathbb{N}^m \quad \overrightarrow{y} \in \mathbb{N}^n}{\phi^{m+n}(p)(\overrightarrow{x} \oplus \overrightarrow{y}) = \phi^m(s_n^m(p, \overrightarrow{x}))(\overrightarrow{y})}}$$

where $\oplus$ is the associative concatenation operation – tclrassociation pattern – on finite lists of inputs.

[Axiom G] says that every partially-computable function M with n-inputs has at least one index, i.e., there exists $m \in \mathbb{N}$ such that $\phi^n(m) = M$, and that there exists for each M a distinct choice of one of its indexes, $d^n(M)$.

$$\dfrac{n \in \mathbb{N} \quad M \in PC_n}{\mathbb{N} \xrightarrow{\phi^n} PC_n \xrightarrow{d^n} \mathbb{N} \qquad \phi^n(d^n(M)) = M}$$

[Axiom A]
The functions $\phi^n, d^n, d^n(M)$ for $n \in \mathbb{N}, M \in PC_n$, are totally-defined computable functions. Identity and projection functions, together with all functions that may be defined solely in terms of computable functions by composition, pairing, evaluation, and adjunction (currying and uncurrying) are computable functions.

6.2 Recursion Theorems

In the sequel, "K2RT" is short for "Kleene's second recursion theorem," and "RRT" abbreviates "Roger's recursion theorem."

Theorem 12. (K2RT[28, 45]) If $T \in PC_2$, then there exists $R \in PC_1$ such that

$$R = \lambda x \in \mathbb{N}.T(d^1(R), x)\,, \tag{17}$$

which conforms to the **recursion pattern**.

Proof. Assume $T \in PC_2$ and define

$$B := \lambda x \in \mathbb{N}.d^1(\lambda y \in \mathbb{N}.\phi^2(x)(x, y)) \tag{18}$$

$$M := \lambda(x, y) \in \mathbb{N} \times \mathbb{N}.T(B(x), y) \tag{19}$$

Then $\mathbb{N} \xrightarrow{B} \mathbb{N}$ is a totally-defined function, and B is a computable function by **Axiom A**since it is defined solely in terms of computable functions d^1 and ϕ^2. Hence, $B \in TC_1$. Likewise, $\mathbb{N} \times \mathbb{N} \xrightarrow{M} \mathbb{N}_\perp$ is a partially-defined computable function by **Axiom A**, since it is defined solely in terms of computable functions T and B. Define

$$R := \lambda x \in \mathbb{N}.T(B(d^2(M)), x)\,. \tag{20}$$

Again, by **Axiom A**, $\mathbb{N} \xrightarrow{R} \mathbb{N}_\perp$ is a computable function, so $R \in PC_1$. Calculate[3]

$$\begin{aligned} B(d^2(M)) &= d^1(\lambda y \in \mathbb{N}.\phi^2(d^2(M))(d^2(M), y)) \\ &= d^1(\lambda y \in \mathbb{N}.M(d^2(M), y)) \quad \text{by } \textbf{Axiom G} \\ &= d^1(\lambda y \in \mathbb{N}.T(B(d^2(M)), y)) \\ &= d^1(\lambda x \in \mathbb{N}.T(B(d^2(M)), x)) \\ &= d^1(R)\,. \end{aligned}$$

Substitution in Equ.(20) completes the proof. □

Theorem 13. (RRT [43, page 180]) If $f \in TC_1$, then there exists $n \in \mathbb{N}$ such that $\phi^1(n) = \phi^1(f(n))$.

Proof. Define

$$\psi = \lambda u \in \mathbb{N}.(\lambda x \in \mathbb{N}.\phi^1(\phi^1(u)(u))(x))\,. \tag{21}$$

Note that a function is not defined if its argument is not defined. So, if $\phi^1(u)(u)$ is not defined, then $\phi^1(u)(u) = \lambda x \in \mathbb{N}.\phi^1((u)(u))(x)$ is the totally undefined function in PC_1. Nevertheless, $\mathbb{N} \xrightarrow{\psi} PC_1$ is a totally-defined computable function, by **Axiom A**. Since $PC_1 \xrightarrow{d^1} \mathbb{N}$ is also totally-defined, by **Axiom G**, the composition $g := d^1 \circ \psi$ is a totally-defined computable function by **Axiom A**. By **Axiom G**,

$$\phi^1 \circ g = \phi^1 \circ d^1 \circ \psi = \psi\,. \tag{22}$$

If, by hypothesis, $f \in TC_1$, then the composition $f \circ g \in TC_1$ by **Axiom A**. Therefore, by **Axiom E**, there exists $v \in \mathbb{N}$ such that

$$f \circ g = \phi^1(v)\,. \tag{23}$$

[3]Lambda calculus experts recognize β reduction and α conversion.

Thus, $\phi^1(v)(v) \in \mathbb{N}$, and so

$$\begin{aligned}
\phi^1(g(v))(x) &= \psi(v)(x) \quad \text{by (22)} \\
&= \phi^1(\phi^1(v)(v))(x) \quad \text{by (21)} \\
&= \phi^1(f(g(v))(x) \quad \text{by (23)}\,.
\end{aligned}$$

Define $n := g(v)$ to complete the proof. ☐

Theorem 14. (RRT([54, page 18])) If $T \in PC_2$, then there exists $m \in \mathbb{N}$ such that

$$\phi^1(s_1^1(m,m)) = \lambda x \in \mathbb{N}.T(s_1^1(m,m),x)\,.$$

Proof. By **Axiom P**, $s_1^1 \in TC_2$. Define

$$M := \lambda(v,x) \in \mathbb{N} \times \mathbb{N}.T(s_1^1(v,v),x)\,.$$

Then $M \in PC_2$ by **Axiom A**. Hence $M = \phi^2(m)$ for some $m \in \mathbb{N}$ by **Axiom E**. Calculate

$$\begin{aligned}
\phi^1(s_1^1(m,m))(x) &= \phi^2(m)(m,x) \quad \text{by } \textbf{Axiom P} \\
&= M(m,x) \\
&= T(s_1^1(m,m),x) \quad \text{by definition}\,.
\end{aligned}$$

☐

Theorem 15. The following are equivalent:
(RRT)

$$\frac{f \in TC_1}{n \in \mathbb{N} \qquad \dfrac{x \in \mathbb{N}}{\phi^1(n)(x) = \phi^1(f(n))(x)}}$$

(K2RT)

$$\frac{p \in \mathbb{N}}{e \in \mathbb{N} \qquad \dfrac{x \in \mathbb{N}}{\phi^1(e)(x) = \phi^2(p)(e,x)}}$$

Proof. (RRT)⇒(K2RT)

$$\frac{\dfrac{p \in \mathbb{N}}{f := s_1^1(p,\_) \in TC_1}}{e \in \mathbb{N} \qquad \dfrac{x \in \mathbb{N}}{\phi^1(e)(x) = \phi^1(f(e))(x) = \phi^1(s_1^1(p,e))(x) = \phi^2(p)(e,x)}}\ \text{RRT}$$

(K2RT)⇒(RRT)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{f \in TC_1}{\mathbb{N} \xrightarrow{f} \mathbb{N}}\ \text{Axiom E} \qquad \mathbb{N} \xrightarrow{\phi^1} PC_1}{\mathbb{N} \xrightarrow{\phi^1 \circ f} PC_1}}{\mathbb{N} \times \mathbb{N} \xrightarrow{(\phi^1 \circ f)^{\downarrow}} \mathbb{N}_{\perp}}}{p \in \mathbb{N} \qquad (\phi^1 \circ f)^{\downarrow} = \phi^2(p)}\ \text{Axiom E}}{\phi^2(p)(n,x) = (\phi^1 \circ f)^{\downarrow}(n,x) = \phi^1(f(n))(x)}}{e \in \mathbb{N} \qquad \phi^1(e)(x) = \phi^2(p)(e,x) = \phi^1(f(e))(x)}\ \text{K2RT}$$

□

6.3 Consequences

Definition 8. A subset $A \subseteq \mathbb{N}$ is **decidable** if, for any two distinct numbers $x_0, x_1 \in \mathbb{N}$, there exists a totally-defined computable function $h \in TC_1$ such that

$$x \in A \Leftrightarrow h(x) = x_1\,. \tag{24}$$

A subset $X \subseteq PC_1$ is called decidable if $(\phi^1)^{-1}X \subseteq \mathbb{N}$ is decidable.

Rice's theorem is about abstract subsets, but its significance is that it applies to formulable subsets, and in particular to the computable subsets of PC_1:

Theorem 16. *(Rice ([54, page 19]))* If $X \subseteq PC_1$ and $\varnothing \neq X \neq PC_1$ then X is not decidable.

Proof. Define $A := (\phi^1)^{-1}X$ and suppose A is decidable. Since $\varnothing \neq X \neq PC_1$, also $\varnothing \neq A \neq \mathbb{N}$. Hence, there exist $x_1 \in A$, $x_0 \notin A$ and $h \in TC_1$ such that

$$x \in A \Leftrightarrow h(x) = x_0\,. \tag{25}$$

By (RRT), there exists $e \in \mathbb{N}$ such that all told,

$$\begin{gathered}\phi^1(e) = \phi^1(h(e))\\ e \in A \Leftrightarrow h(e) \notin A\\ \phi^1(e) \in X \Leftrightarrow \phi^1(h(e)) \notin X\end{gathered}$$

which is a blatant contradiction. □

Theorem 17. If $\varnothing \neq X \nsubseteq PC_1$, and $f \in X$, $g \in PC_1 - X$, then the function $\mathbb{N} \times \mathbb{N} \xrightarrow{Q} \mathbb{N}_{\perp}$ defined by

$$Q(x,z) = \begin{cases} f(z) \text{ if } \phi^1(x) \notin X \\ g(z) \text{ if } \phi^1(x) \in X \end{cases}$$

is not computable, i.e., not in PC_2.

Proof. If $Q \in PC_2$, then by (K2RT) there exists $e \in \mathbb{N}$ such that $\phi^1(e)(z) = Q(e,z)$ for all $z \in \mathbb{N}$. Therefore, if $\phi^1(e) \in X$, then $\phi^1(e)(z) = Q(e,z) = g(z)$, so $\phi^1(e) = g \notin X$, which is a contradiction. Likewise, there is a contradiction if $\phi^1(e) \notin X$. Hence, $Q \in PC_2$ is impossible. $\square$

7 Conclusion

Notation patterns abound in mathematics from elementary to advanced contexts. **Adjunction patterns** and **recursion patterns** express advanced circles of ideas containing large areas of mathematics. Perhaps the area of such a circle is proportional to the symmetry of the notation. Then again, the reader might say there is nothing special here, that of course the **adjunction pattern** and the **recursion pattern** are inevitable mathematical notations – that there could not be any mathematics without them.

Or, maybe there is something essentially *human* about these patterns, that maybe some extra-terrestrially evolved intelligence is equally sophisticated intellectually, but deploys completely different patterns to achieve its own understanding of mathematics and nature in those terms. If that is possible, one may ask, what is it about biological evolution of human beings that they are equipped with minds that find these patterns so appealing?

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